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A Contribution to Best Approximation in the L^2 Norm

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The set of all functions $f(Z)$ which are holomorphic in the open unit disk B and for which

$$\int_B \int |f(z)|^2 \, dx \, dy < \infty,$$

will be denoted by $L^2(B)$.

Set, for any $f \in L^2(B)$, and for $n = 0, 1, \ldots,$

$$\Delta_n(f) = \min_{a_i} \left[\iint_B |f(Z) - a_0 - a_1 Z - \cdots a_n Z^n|^2 \, dx \, dy \right]^{1/2}. \tag{1}$$

Let $f(x)$ be a real continuous function on $[-1, 1]$ and $\omega(x)$ a real non-negative Riemann integrable function on $[-1, 1]$; then set

$$E_n^{(2)}(f, \omega) = E_n^{(2)}(f) = \min_{p \in \pi_n} \left[\int_{-1}^{1} |f(x) - p(x)|^2 \, \omega(x) \, dx \right]^{1/2},$$
$$n = 0, 1, 2, \ldots, \tag{2}$$

where π_n denotes the class of all real polynomials of degree at most n. Further, let

$$E_n(f) = \min_{p \in \pi_n} \|f(x) - p(x)\|, \qquad n = 0, 1, \ldots, \tag{3}$$

where $\| \ \|$ is the uniform norm on $[-1, 1]$.

Recently, it has been studied [4] how $E_n(f)$, for f which is a restriction of an entire function, is related to the order and type of the function. In [5], we have investigated how the Taylor coefficients of an entire function are related to the error in best approximations (as in (3)). In [6], we have discussed how $E_n(f)$ is related to $M_n(f) \equiv \max_{-1 \leqslant x \leqslant 1} |f^{(n)}(x)|$ for entire functions.

We are here interested in knowing how $\Delta_n(f)$ is related to the order and type of f, assumed to be entire. We also investigate how $\Delta_n(f)$ is related to the $(n+1)$st coefficient of the Taylor expansion of f. Further, we show how $E_n^{(2)}(f)$ is related to the order and type of the entire function. Finally, we obtain an asymptotic relation between $E_n(f)$ and $E_n^{(2)}(f)$ for a sequence of values of n.

THEOREM 1. *A necessary and sufficient condition for $f(Z) \in L^2(B)$ to be an entire function is that*

$$\lim_{n\to\infty} [\Delta_n(f)]^{1/n} = 0. \tag{6}$$

Proof. Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k \in L^2(B)$; it is known [3, p. 333] that

$$[\Delta_n(f)]^2 = \sum_{k=n+1}^{\infty} \frac{\pi}{k+1} \mid a_k \mid^2. \tag{7}$$

Now let us suppose that $f(Z)$ is entire; then

$$\lim_{k\to\infty} \mid a_k \mid^{1/k} = 0. \tag{8}$$

One has from (7) and (8), for any $\epsilon > 0$ and for all $n \geqslant n_0(\epsilon)$, (9)

$$[\Delta_n(f)]^2 \leqslant \sum_{k=n+1}^{\infty} \frac{\pi}{k+1} \epsilon^{2k}$$
$$< \frac{\pi \epsilon^{2(n+1)}}{(n+2)\cdot(1-\epsilon^2)}, \tag{10}$$

ϵ being arbitrary, (6) follows from (10).

If (6) is true, then since

$$[\Delta_n(f)]^2 \geqslant \frac{\pi}{(n+2)} \mid a_{n+1} \mid^2, \tag{11}$$

we have $\lim_{k\to\infty} \mid a_k \mid^{1/k} = 0$, and hence f is entire.

Remark. The first part of the theorem has been stated without proof in [3, p. 334].

THEOREM 2. *If $f(Z) \in L^2(B)$, then $f(Z)$ is an entire function of order $\rho (0 < \rho < \infty)$ if, and only if,*

$$\limsup_{n\to\infty} \frac{n \log n}{\log[1/\Delta_n(f)]} = \rho. \tag{12}$$

Proof. Assume (12). Then (6) follows, and, hence, f is an entire function. Suppose that $f(Z) = \sum_{j=0}^{\infty} a_j z^j$ is of order α. Then [2, p. 9]

$$\alpha = \limsup_{n\to\infty} \frac{n \log n}{\log[1/|a_n|]}. \tag{13}$$

Given any $\epsilon > 0$, we have from (13) for all $k \geqslant k_0(\epsilon)$,

$$|a_k| \leqslant k^{-k/(\alpha+\epsilon)}. \tag{14}$$

One obtains from (7) and (14) with a little manipulation:

$$\begin{aligned} [\Delta_n(f)]^2 &= \sum_{k=n+1}^{\infty} \frac{\pi}{(k+1)} |a_k|^2 \leqslant \frac{\pi}{(n+2)} \sum_{k=n+1}^{\infty} \frac{1}{k^{2k/(\alpha+\epsilon)}} \\ &\leqslant \frac{\pi}{(n+2)} \frac{1}{(n+1)^{2(n+1)/(\alpha+\epsilon)}} \left[\sum_{j=0}^{\infty} (n+2)^{-2j/(\alpha+\epsilon)}\right] \\ &\leqslant \frac{\pi}{(n+1)^{2(n+1)/(\alpha+\epsilon)}} [1-(n+2)^{-2/(\alpha+\epsilon)}]^{-1}. \end{aligned} \tag{15}$$

ϵ being arbitrary, we have, from (12) and (15),

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log[1/\Delta_n(f)]} \leqslant \alpha.$$

From (7) and (13)

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log[1/\Delta_n(f)]} \geqslant \limsup_{n\to\infty} \frac{n \log n}{\log |1/a_n|} = \alpha,$$

hence $\rho = \alpha$.

The converse part of the theorem can again be derived from (7) and (13).

THEOREM 3. *If $f \in L^2(B)$, then $f(Z)$ is an entire function of order $\rho (0 < \rho < \infty)$ and type $\tau (0 < \tau < \infty)$ if and only if*

$$\limsup_{n\to\infty} \frac{n}{\rho e} [\Delta_n(f)]^{\rho/n} = \tau. \tag{16}$$

Proof. (16) implies (12) which, in turn, implies that $f(Z)$ is an entire function of order ρ. It is known [2, p. 8] that the type of f is

$$\beta = \limsup_{n\to\infty} \frac{n}{\rho e} |a_n|^{\rho/n} (0 < \beta < \infty). \tag{17}$$

From (17) we have, for all $k \geqslant k_0(\epsilon)$,

$$| a_k | \leqslant \frac{(\rho e(\beta + \epsilon))^{k/\rho}}{k}. \tag{18}$$

From (7) and (18) we can deduce, as in the proof of Theorem 2, that

$$\tau = \limsup_{n\to\infty} \frac{n}{\rho e} [\Delta_n(f)]^{\rho/n} \leqslant \beta.$$

Also, from (11) and (17)

$$\tau = \limsup_{n\to\infty} \frac{n}{\rho e} [\Delta_n(f)]^{\rho/n} \geqslant \limsup_{n\to\infty} \frac{n}{\rho e} | a_n |^{\rho/n} = \beta.$$

Hence $\tau = \beta$.

If $f(Z)$ is entire of order $\rho (0 < \rho < \infty)$ and type $\tau (0 < \tau < \infty)$, then we can show easily, using (7) and (18), that (16) holds.

THEOREM 4. *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$ be an entire transcendental function. Then there exists a sequence of integers $0 \leqslant n_1 < n_2 < \cdots < n_p < \cdots$ for which all a_{n_p+1} are $\neq 0$ and*

$$\lim_{p\to\infty} \frac{(n_p + 2)^{1/2} \Delta_{n_p}(f)}{| a_{n_p+1} |} = \pi^{1/2}. \tag{19}$$

Proof. Since $f(Z)$ is entire,

$$\lim_{k\to\infty} | a_k |^{1/k} = 0. \tag{20}$$

Let $(\epsilon_p)_{p=1}^{\infty}$ be an arbitrary sequence of positive numbers <1, with $\epsilon_p \to 0$. There are [3a, pp. 128–129], for $p = 1, 2,\ldots$, infinitely many integers $n_p \geqslant 0$ for which

$$| a_{n_p+j} | \leqslant | a_{n_p+1} | \epsilon_p^{j-1} \qquad (j = 1, 2,\ldots). \tag{21}$$

For each $p \geqslant 1$ we choose such an n_p so that $0 \leqslant n_1 < n_2 < \cdots$. Then, for every $p \geqslant 1$, by (7),

$$\begin{aligned} \left(\frac{\pi}{n_p + 2}\right)^{1/2} | a_{n_p+1} | < \Delta_{n_p}(f) &< \left(\frac{\pi}{n_p + 2}\right)^{1/2} \left[\sum_{j=1}^{\infty} | a_{n_p+j} |^2\right]^{1/2} \\ &\leqslant \frac{| a_{n_p+1} | \pi^{1/2}}{(n_p + 2)^{1/2}} \left(\sum_{j=0}^{\infty} \epsilon_p^{2j}\right)^{1/2} = \frac{| a_{n_p+1} | \pi^{1/2}}{(n_p + 2)^{1/2} (1 - \epsilon_p^2)^{1/2}} \end{aligned} \tag{22}$$

from which (19) follows.

THEOREM 5. *Let $f(x)$ be a real continuous function on $[-1, 1]$. A necessary and sufficient condition for $f(x)$ to be the restriction to $[-1, 1]$ of an entire function is that*

$$\lim_{n\to\infty} [E_n^{(2)}(f)]^{1/n} = 0. \tag{23}$$

Proof. Let $A_0/2 + \sum_{k=1}^{\infty} A_k T_k(x)$ be the Fourier–Chebyshev series of $f(x)$. Then

$$[E_n^{(2)}(f)]^2 = \frac{\pi}{2} \sum_{k=n+1}^{\infty} A_k^2. \tag{24}$$

Further, it is known [1, p. 111] that

$$| A_{n+1} | + | A_{n+2} | + | A_{n+3} | + \cdots \geqslant E_n \geqslant [\tfrac{1}{2}(A_{n+1}^2 + A_{n+2}^2 + \cdots +)]^{1/2}; \tag{25}$$

hence,

$$\pi E_n^2 \geqslant [E_n^{(2)}]^2. \tag{26}$$

If $f(x)$ is the restriction to $[-1, 1]$ of an entire function, then [1, p. 111]

$$\lim_{n\to\infty} E_n^{1/n} = 0, \tag{27}$$

which implies (23).

On the other hand, if $\lim_{n\to\infty}[E_n^{(2)}]^{1/n} = 0$, then from (24) we have

$$\lim_{k\to\infty} | A_k |^{1/k} = 0, \tag{29}$$

which implies, by (25), that $\lim_{n\to\infty}[E_n(f)]^{1/n} = 0$; this, in turn, implies that $f(x)$ is the restriction to $[-1, 1]$ of an entire function [1, p. 113].

THEOREM 6. *If $f(x)$ is a real continuous function on $[-1, 1]$, then it is the restriction to $[-1, 1]$ of an entire function of order $\rho (0 < \rho < \infty)$ if and only if*

$$\limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n^{(2)}]} = \rho. \tag{30}$$

Proof. (30) implies (23), namely, that $f(x)$ is the restriction to $[-1, 1]$ of an entire function, say of order α. Now it follows from (26) and Theorem 1 of [7] that

$$\alpha = \limsup_{n\to\infty} \frac{n \log n}{[\log 1/E_n(f)]} \geqslant \limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n^{(2)}]} = \rho. \tag{31}$$

From (30) and (24), for a given $\epsilon > 0$ and for all sufficiently large n,

$$\frac{\pi}{2} A_{n+1}^2 \leqslant [E_n^{(2)}]^2 \leqslant n^{-2n/(\rho+\epsilon)}. \tag{32}$$

Now it follows, with a little manipulation, from (25) and (32), that

$$\alpha = \limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n(f)]} \leqslant \rho, \tag{33}$$

hence, $\alpha = \rho$. Conversely, suppose $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ. Then we have from (26) and Theorem 1 of [7],

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n(f)]} \geqslant \limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n^{(2)}]}. \tag{34}$$

As in the first part of the proof we obtain also the reverse inequality for ρ. Hence (30).

THEOREM 7. *Let $f(x)$ be a real continuous function on* $[-1, 1]$. *A necessary and sufficient condition for $f(x)$ to be the restriction to* $[-1, 1]$ *of an entire function of order $\rho(0 < \rho < \infty)$ and type $\tau(0 < \tau < \infty)$ is that*

$$\limsup_{n\to\infty} \frac{n}{\rho e} [E_n^{(2)}]^{\rho/n} = \frac{\tau}{2^\rho}. \tag{35}$$

Proof. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order $\rho(0 < \rho < \infty)$ and type $\tau(0 < \tau < \infty)$, then, as is known [4, Theorem 3]

$$\limsup_{n\to\infty} \frac{n}{\rho e} [E_n(f)]^{\rho/n} = \frac{\tau}{2^\rho}. \tag{36}$$

One has from (26) and (36),

$$\frac{\tau}{2^\rho} \geqslant \limsup_{n\to\infty} \frac{n}{\rho e} [E_n^{(2)}]^{\rho/n}. \tag{37}$$

One can prove the reverse inequality for $\tau/2^\rho$ as in the proof of Theorem 6. We omit the proof of the second half of the theorem.

THEOREM 8. *If $A_0/2 + A_1T_1(x) + A_2T_2(x) + \cdots + A_nT_n(x) + \cdots$ is the Fourier–Chebyshev series expansion of a real function $f(x)$, defined on* $[-1, 1]$, *which is the restriction of an entire transcendental function, then there exists a* *sequence of integers* $0 \leqslant n_1 < n_2 < \cdots < n_p < \cdots$ *for which all* A_{n_p+1} *are* $\neq 0$ *and*

$$\lim_{p\to\infty} \frac{\sqrt{2}\, E_{n_p}^{(2)}}{|A_{n_p+1}|} = \pi^{1/2}. \tag{38}$$

Proof. The proof follows the same lines as that of Theorem 4, with the difference that instead of using (7) and (20) we use (24) and (29).

THEOREM 9. *If* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function, then there exists a sequence of integers* $0 \leqslant n_1 < n_2 < \cdots < n_p < \cdots$ *for which*

$$\sqrt{2}\, E_{n_p}^{(2)} \sim \pi^{1/2} E_{n_p}. \tag{39}$$

Proof. We know from Theorem 8 that for some such sequence $(n_p)_{p=1}^{\infty}$,

$$\sqrt{2}\, E_{n_p}^{(2)} \sim \pi^{1/2} |A_{n_p+1}|. \tag{40}$$

It is also known [1, (167), p. 115] that for some such sequence $(n_p')_{p=1}^{\infty}$,

$$E_{n_p'} \sim |A_{n_p'+1}|. \tag{41}$$

If we can show that (40) and (41) hold for the same sequence, then (39) will follow. In proving (40), we used the fact that

$$\lim_{k\to\infty} |A_k|^{1/k} = 0, \tag{29}$$

from which follows that, given $\epsilon > 0$, for infinitely many integers $n \geqslant 0$ we have

$$|A_{n+j}| \leqslant |A_{n+1}|\, \epsilon^{j-1} \qquad (j = 1, 2, \ldots). \tag{42}$$

Bernstein used [1, (168), p. 115] only (29) and (42) to prove (41), hence one can choose $n_p \equiv n_p'$.

REFERENCES

1. S. N. BERNSTEIN, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réele," Gauthiers-Villars, Paris, 1926; Chelsea, New York, 1970.
2. R. P. BOAS, "Entire Functions," Academic Press, New York, 1954.
3. P. J. DAVIS, "Interpolation and Approximation," 2nd ed., Blaisdell Publishing Company, New York, 1965.

3a. P. ERDÖS AND A. RÉNYI, On the number of zeros of successive derivatives of analytic functions, *Acta Math. Acad. Sci. Hungar.* 7 (1956), 125–144.

4. A. R. REDDY, Approximation of an entire function, *J. Approximation Theory* **3** (1970), 128–137.
5. A. R. REDDY, Best polynomial approximation to certain entire functions; *J. Approximation Theory* **5** (1972), 97–112.
6. A. R. REDDY, Best polynomial approximation to certain entire functions II, to appear.
7. R. S. VARGA, On an extension of a result of S. N. Bernstein, *J. Approximation Theory* **1** (1968), 176–179.